Recent developments in non-Abelian T-duality in string theory *

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Synopsis

We briefly review the essential points of our recent work in non-Abelian T-duality. In particular, we show how non-abelian T-duals can effectively describe infinitely high spin sectors of a parent theory and how to implement the transformation in the presence of non-vanishing Ramond fields in type-II supergravity.

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1 Prolegomena

Abelian T-duality was originally formulated in a path integral approach in which central rôle played the isometry group $G = U(1)$ of the background one chooses to dualize [1]. When this group is non-Abelian one may follow a similar path to naturally arrive at the notion of non-Abelian T-duality [2]. However, the similarities between the two cases stop here. In particular:

1. Unlike the Abelian case, the non-Abelian T-duality transformation is non-invertible in the standard path integral formulation since the isometries are no longer present in the T-dual background (however, see comments at the end of this note).

2. For compact commuting isometries one may argue that T-duality is actually a true symmetry of string theory. There is no analogous statement for the non-Abelian cases.

3. Even for compact groups, the variables of the T-dual background are generically non-compact.

4. The formulation of T-duality in the presence of Ramond fluxes presents technical difficulties. In the Abelian case the unique dimensional reduction to nine dimensions of the type-II supergravities provided for the transformation rules [3]. This possibility hasn't been explored for non-Abelian T-duality.

In this note we summarize the essential points of recent developments in the subject.

2 Pure NS backgrounds

This section is based mainly on [4] and on general techniques developed in [5]. Consider a pure NS background described by a metric $G_{\mu\nu}$, an antisymmetric tensor $B_{\mu\nu}$ and a dilaton Φ. These can couple in a classical two-dimensional σ-model action for the target space variables X^μ. Let's denote this action by $S(X)$ and the corresponding theory by $\mathcal{C}$. We will also assume that there is an isometry group G leaving the action invariant with the variables X^μ transforming accordingly. We gauge a subgroup $H \subset G$ by introducing, in a Buscher-like approach, gauge fields $A_\pm \in \mathcal{L}(H)$ and a Lagrange multiplier term for

their field strength $F_{\pm}$. The corresponding action is

$$S_{\text{T-dual}}(X, v, A_{\pm}) = S_{\text{g}}(X, A_{\pm}) - i \int d^2\sigma \text{Tr}(vF_{+-}) \ , \qquad S_{\text{g}}(X, 0) = S(X) \ . \tag{2.1}$$

This action should be invariant under $A_{\pm} \rightarrow \Lambda^{-1}(A_{\pm} - \partial_{\pm})\Lambda$ and $v \rightarrow \Lambda^{-1}v\Lambda$, together with the transformation of the X^{μ}'s. The gauge fields enter at most quadratically and non-dynamically in $S_{\text{g}}(X, A_{\pm})$. Integrating them out gives the T-dual σ-model with transformed background fields. We should also gauge fix $\dim(H)$ among the parameters X^{μ} and the Lagrange multipliers in v. The maximum number of entries in v that can be gauge fixed cannot exceed $\dim(H) - \text{rank}(H)$.

A non-Abelian T-dual ia a generically non-compact manifold even for compact isometries. The corresponding σ-model is the result of a delicate limit taken in some parent theory. It effectively describes sector(s) of infinitely high quantum numbers with a simultaneous stretching of coordinates in the corresponding parent background. To see that consider adding to $S_{\text{g}}(X, A_{\pm})$, not the Lagrange multiplier term, but the gauged WZW action $I(h, A_{\pm})$ at level ℓ for the group element $h \in H$. Expanding infinitesimally as $h = \mathbb{I} + iv/\ell + \cdots$ and taking the limit $\ell \rightarrow \infty$ one finds that $I(h, A_{\pm})$ precisely reproduces the Lagrange multiplier term. Hence, one may think of the non-Abelian dual of the original theory $\mathcal{C}$ with action $S(X)$ as the limit of the gauged tensor product theory $\mathcal{C} \times H_{\ell}$, when $\ell \rightarrow \infty$. The advantage of this point of view is that the original theory may be better suited to study before the limit is taken. Also, the non-compactness of the dual variables in v is naturally explained.

The above classical statement can be promoted at the level of the quantum states of the theory. Consider as the simplest example the non-Abelian T-dual of the $SU(2)$ WZW model with respect to $SU(2)$ acting vectorially. In this case $\mathcal{C}$ is a current algebra theory and $S(X)$ is the associated WZW model action. The T-dual background has zero antisymmetric tensor. The metric and dilaton read

$$\begin{aligned} ds^2 &= d\psi^2 + \frac{\cos^2\psi}{x_3^2} dx_1^2 + \frac{(x_3 dx_3 + (\sin\psi\cos\psi + x_1 + \psi)dx_1)^2}{x_3^2\cos^2\psi} \ , \\ \Phi &= -\ln(x_3\cos\psi) \ , \end{aligned} \tag{2.2}$$

where ψ is periodic and x_1, x_3 are non-compact, a background possessing no isometries. It describes the infinitely large spin sector of the $SU(2)_{k_1} \times SU(2)_{k_2}/SU(2)_{k_1+k_2}$ coset

CFT model. In Physics it is important to be able to solve the field equations in a given background especially the scalar wave equation. Doing that by traditional methods is hopeless given the complexity of the background. Making use of the underlying CFT, the general state can be written as a multiple sum involving the Clebsch–Gordan coefficients for a state $|j, m\rangle$ in the diagonal $SU(2)$ composed from states $|j_1, m_1\rangle|j_2, m_2\rangle$ in $SU(2) \times SU(2)$ for the left and the right sectors separately and the Wigner's d-functions, is such a way that a singlet of the diagonal $SU(2)_L \times SU(2)_R$ is formed. The eigenvalues are (for $k_{1,2} \gg 1$)

$$E^j_{j_1,j_2} = \frac{j_1(j_1+1)}{k_1} + \frac{j_2(j_2+1)}{k_2} - \frac{j(j+1)}{k_1+k_2} \,. \tag{2.3}$$

To illustrate how the high spin limit is taken assume that one of the spins is extremely large, i.e. $j_1, j \gg 1$ and j_2 =finite. In this limit, the eigenvalues become infinite unless the level k_1 becomes large as well and proportional to j. Specifically, let $j_1 = j - n$ and $k_1 = k_2 j/\delta$, where $|n| \leqslant j_2$ and $\delta \in \mathbb{R}^+$. Then

$$E_{j_2,n,\delta} = \lim_{j\to\infty} E^j_{j_1,j_2} = \frac{j_2(j_2+1)}{k_2} + \frac{\delta - 2n}{k_2}\,\delta \,. \tag{2.4}$$

It turns out that in the $k_1 \to \infty$ limit the background of the coset model becomes that in (2.2). Also, the solutions of the scalar wave equation can be obtain from a delicate limit of the corresponding solutions of the coset model. For example, the states with $j_2 = 1/2$ are

$$\Psi_{1/2,\pm 1/2,\delta} = \pm\frac{\beta_3}{\delta v_3}\,\cos 2\delta v_3 + \frac{2\delta\beta_0 v_3 \mp \beta_3}{2\delta^2 v_3^2}\,\sin 2\delta v_3 \,, \tag{2.5}$$

where v_3, β_0 and β_3 are functions of the variables x_1, x_2 and ψ. We couldn't have constructed this solution, let alone one for general spin j, by directly solving the scalar wave equation for the background (2.2).

3 Non-trivial RR backgrounds

This section is based on [6, 7]. In type-II supergravity it is necessary to know, in addition to the NS fields, how Ramond fluxes transform under T-duality. The left and right world sheet derivatives transform differently under T-duality and this defines two orthonormal

frames related by a Lorentz transformation matrix Λ. The induced action on spinors is given by a matrix Ω obtained by

$$\Omega^{-1}\Gamma^i\Omega = \Lambda^i{}_j\Gamma^j \, . \tag{3.1}$$

The RR-fields are combined into a bi-spinor according to which type-II supergravity they belong to as

$$\text{IIB}: \quad P = \frac{e^\Phi}{2}\sum_{n=0}^{4}\frac{\not{F}_{2n+1}}{(2n+1)!} \, , \qquad \text{(massive) IIA}: \quad P = \frac{e^\Phi}{2}\sum_{n=0}^{5}\frac{\not{F}_{2n}}{(2n)!} \, , \tag{3.2}$$

with $\not{F}_p = \Gamma^{\mu_1\cdots\mu_p}F_{\mu_1\cdots\mu_p}$. and where we have used the democratic formulation of type-II supergravities where all forms up to order ten appear. The fluxes transform under T-duality according to

$$\hat{P} = P\Omega^{-1} \, , \tag{3.3}$$

where we have denoted by a hat the bi-spinor obtained after the duality. The details of the matrix Ω depend on the case of interest. For comparison, for Abelian T-duality this is simply given by $\Omega = \Gamma_{11}\Gamma_1$ [8], where 1 labels the isometry direction and Γ_{11} the product of all Gamma matrices. In the Abelian case we flip between type-IIA and type-IIB, but in non-Abelian cases we might change or stay within the same theory.

Many interesting supergravity backgrounds have as an essential part group or coset manifolds. Hence, we concentrate on Principle Chiral-type models which can cover both cases as we will see. Consider an group element $g \in G$ and the components of the left invariant Maurer–Cartan forms $L^a_\mu = -i\,\mathrm{Tr}(t^a g^{-1}\partial_\mu g)$. The representation matrices t^a obey the Lie algebra with structure constants $f^{ab}{}_c$. The most general σ-model invariant under the global symmetry $g \to g_0 g$, with $g_0 \in G$, is (we ignore spectator fields)

$$S = \frac{1}{2}\int d^2\sigma\, E_{ab}L^a_+L^b_- \, , \qquad L^a_\pm = L^a_\mu\partial_\pm X^\mu \, . \tag{3.4}$$

Consider first the case in which a and b run over the whole group. Then E is a $\dim(G)$ square invertible constant matrix. It turns out that the T-dual σ-model with respect to the

full G symmetry group is

$$\tilde{S} = \frac{1}{2}\int d^2\sigma\,(M^{-1})^{ab}\partial_+ v_a \partial_- v_b\ , \quad M_{ab} = E_{ab} + f_{ab}\ , \qquad f_{ab} = f_{ab}{}^c v_c \tag{3.5}$$

and the induced dilaton is $\Phi = -\frac{1}{2}\ln\det M$. The variables of the T-dual model are the Lagrange multipliers v_a since it is possible to gauged fix the group element g to unity. This is possible since the left sided group action acts with no isotropy. The world-sheet derivatives transform as

$$L_+^a = (M^{-1})^{ba}\partial_+ v_b\ , \qquad L_-^a = -(M^{-1})^{ab}\partial_- v_b\ . \tag{3.6}$$

Denoting by $\eta = \kappa^T\kappa$ the symmetric part of E, the frame relating Lorentz transformation is

$$\Lambda = -\kappa M^{-1T} M \kappa^{-1} \quad \Longrightarrow \quad \Omega = e^{\frac{1}{2}\tilde{f}_{ab}\Gamma^{ab}} \prod_{i=1}^{\dim(G)} (\Gamma_{11}\Gamma_i)\ , \quad \tilde{f} = \kappa^{-1T}(S+f)\kappa^{-1}\ . \tag{3.7}$$

Clearly if the dimensionality of the duality group is even then we stay in the same type-II supergravity theory, otherwise we flip from (massive) type–IIA supergravity to type–IIB and vise versa.

To extend the discussion for coset G/H σ-models, we split the index $a = (i,\alpha)$, where the indices i and α belong to the subgroup $H \in G$ and the coset G/H, respectively. In (3.4) we consider, instead of E, a matrix E_0 with coset indices only, a restriction requiring that E_0 is G-invariant. For coset models the group acts with isotropy and one has to gauge fix $\dim(H)$ variables among the Lagrange multipliers v_a. Denoting the remaining variables by x_α, we define the $\dim(G/H)$-dimensional square matrices $N_\pm$ associated with the orthonormal frames, from the relations

$$L_+^\alpha = (M^{-1})^{b\alpha}\partial_+ v_b = N_+^{\alpha\beta}\partial_+ x_\beta\ , \qquad L_-^\alpha = -(M^{-1})^{\alpha b}\partial_- v_b = N_-^{\alpha\beta}\partial_- x_\beta\ . \tag{3.8}$$

The Lorentz transformation that relates them is given by (we write $E_0 = \kappa_0^T\kappa_0$)

$$\Lambda = \kappa_0 N_+ N_-^{-1}\kappa_0^{-1}\ . \tag{3.9}$$

As an example consider within the type-IIB supergravity the $AdS_3 \times S^3 \times T^4$ geometry

arising in the near horizon of the D1-D5 brane system. This is supported by an F_3 flux given by the sum of the volume forms of the two group spaces. The presence of S^3 indicates an $SO(4) \simeq SU(2)_L \times SU(2)_R$ isometry group. Hence we may T-dualize with respect to the full $SO(4)$ or with respect to $SU(2)_L$. Consider the latter situation first. One obtains that the fields of the NS sector of dual model are

$$ds^2 = ds^2(\mathrm{AdS}_3) + dr^2 + \frac{r^2}{1+r^2} d\Omega_2^2 + ds^2(T^4) \,,$$
$$B = \frac{r^3}{1+r^2} d\mathrm{Vol}(S^2) \,, \qquad \Phi = -\frac{1}{2} \ln(1+r^2) \,. \tag{3.10}$$

The background corresponds to a smooth space, due to the fact that the isometry acts with no isotropy. In this case the Lorentz transformation is given by (below $r^2 = x_i x_i$)

$$\Lambda_{ij} = \frac{r^2-1}{r^2+1}\,\delta_{ij} - \frac{2}{r^2+1}(x_i x_j + \epsilon_{ijk} x_k) \quad \Longrightarrow \quad \Omega = \Gamma_{11}\left(\frac{\Gamma_{123} + \mathbf{x}\cdot\mathbf{\Gamma}}{\sqrt{1+r^2}}\right) \,, \tag{3.11}$$

where $1, 2$ and 3 refer to the directions along the non-Abelian T-dual of S^3. Using (3.3) we obtain the fluxes

$$F_0 = 1 \,, \quad F_2 = \frac{r^3}{1+r^2}\, d\mathrm{Vol}(S^2) \,, \quad F_4 = -r dr \wedge d\mathrm{Vol}(AdS_3) + d\mathrm{Vol}(T^4) \,. \tag{3.12}$$

This is a solution of massive IIA supergravity and has the residual $SU(2)_R \subset SO(4)$ symmetry.

In order to dualize with respect to the full $SO(4)$ symmetry we follow the procedure outlined above since this group acts with isotropy on the Lagrange multipliers. We compute the two orthonormal frames corresponding to (3.8) and the associated Lorentz transformation using (3.9) (with $\kappa_0 = \mathbb{I}$). We find that

$$N_+ = \frac{1}{x_1 x_3}\begin{pmatrix} 0 & x_2 & x_3 \\ 0 & x_2^2 - x_1^2 & x_2 x_3 \\ x_1 x_3 & x_2 x_3 & x_3^2 \end{pmatrix} \,, \quad \Lambda = \mathrm{diag}(1,-1,-1) \,, \quad \Omega = -\Gamma_2\Gamma_3 \,. \tag{3.13}$$

Hence, Ω, is as if we had two successive Abelian T-dualities. The NS two-form vanishes and the dilaton is computed to be $\Phi = -\ln(x_1 x_3)$. Then we compute the Ramond fluxes as

$$F_1 = 2(x_2 dx_3 + x_3 dx_3) \,, \qquad F_5 = (1+\star)(F_1 \wedge d\mathrm{Vol}(T^4)) \,. \tag{3.14}$$

This T-dual background is a solution of type-IIB supergravity. The singularity for $x_1 x_3 = 0$ is associated with the fact that the group acts with isotropy.

Finally note that, a manifestly T-duality invariant path integral formulation of non-Abelian T-duality is known in the broader context of Poisson-Lie T-duality by doubling the coordinates in a first order σ-model action [9]. It is interesting to embed this in string theory including the RR-sector fields.

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